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1987 J. Phys. A: Math. Gen. 20 1973

(http://iopscience.iop.org/0305-4470/20/8/017)

J. Phys. A: Math. Gen. 20 (1987) 1973-1988. Printed in the UK

Exact solutions of the Broadwell model in 1+1 dimensions

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Received 1 July 1986, in final form 11 September 1986

Abstract. We study the one spatial dimensional, 6-velocity Broadwell model with four identical densities and three independent ones. We determine 'solitons' (one-dimensional shock wave solutions) and 'bisolitons' (two-dimensional, space plus time solutions) which are rational fractions with one or two exponential variables.

We obtain three classes of positive exact solutions in 1+1 dimensions (space x, time t). The first one is periodic in the space variable and for large time the solutions correspond to propagating damped linear waves. The second is positive only along one semi x axis while the third, positive along the whole x axis, represents non-planar damped shock waves.

Using the same tools in a companion paper, for the discrete 2-velocity models, we obtain in a two-dimensional space the first two classes of solutions mentioned above. This suggests that, for the discrete Boltzmann models, general methods exist for the determination of non-trivial exact solutions.

1. Introduction

It is generally thought that the study of discrete Boltzmann models may provide useful hints for the present problems in kinetic theory. The most popular discrete model is the Broadwell (1964) one. The general Broadwell model is a discrete 6-velocity model of the Boltzmann equation (BE) in three spatial dimensions. In general a simplified one-dimensional version is studied (which is the one originally introduced by Broadwell for the determination of explicit planar shock solutions). Let us call V and W the densities for particles with velocities $(\pm 1, 0, 0)$ and assume the same density for those with velocities $(0, \pm 1, 0)$ and $(0, 0, \pm 1)$. In only one spatial x dimension, the resulting equations are:

$$V_t + V_x = W_t - W_x = -2Z_t = Z^2 - VW. \qquad (1.1)$$

The H-theorem is satisfied and there exist two independent linear differential relations which correspond to the conservation of mass ($N = V + W + 4Z$) and momentum (current $J = V - W$):

$$N_t + J_x = 0 \qquad J_t + V_x + W_x = 0. \qquad (1.2)$$

Besides its original interest in the shock-wave problem, this model has been thoroughly studied, as a laboratory tool of the discrete BE, for the proofs of global existence, uniqueness and boundedness properties of the solutions (Nishida and Miura 1974, Crandall and Tartar 1976, Inoue and Nishida 1976, Caflish and Papanicolaou

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1979, Tartar 1980, Illner 1984, Beale 1985). These results were extended by Cabannes to his 14-velocity model (Cabannes 1978, Gatignol 1975).

Our aim is to determine non-trivial classes of physically acceptable solutions of (1.1) in $1+1$ dimensions (space x, time t). For obvious reasons, the mathematical results were obtained for 'smooth' initial data. For instance the densities must be integrable when $|x|\to\infty$ and here in general the explicit solutions will not satisfy this requirement. We find essentially two classes of two-dimensional solutions: either periodic in x or positive and non-periodic along the full x axis (a larger class contains positive solutions along the semi x axis). Once more we recall (a fact sometimes forgotten) that the primary motivation of the Broadwell discrete model was the construction of a simplified version of the BE leading to explicit planar (one-dimensional) shock solutions. It seems natural to investigate in higher dimensions (two for (1.1)) whether other explicit solutions could represent physically relevant situations (in particular if there exist generalisations of the planar shock solutions). Here the periodic solutions, for large times, represent damped propagating planar waves. They could correspond to damped sound waves, but the current J has in general a non-vanishing asymptotic limit when t is infinite. For the particular solutions for which this limit is zero then the waves are non-propagating with time. The positive two-dimensional solutions on the full x axis are the non-planar generalisations of the planar shock profiles. The shocks are damped with increasing time and the densities relax towards Maxwellian equilibrium states.

In a companion paper (Cornille 1987), with the same tools as here, we study two-dimensional solutions of the 2-velocity discrete models (Illner 1979). We find damped sound wave periodic solutions but not non-periodic solutions positive along the whole x axis (the positivity difficulty is already present for the one-dimensional shock waves). Different positivity properties exist for 2- and 3-discrete models. However they can be studied with the same algebraic method.

Here we face (1.1) as a genuine non-integrable equation. We define 'solitons' and 'bisolitons' as rational solutions with only one or two exponential variables $\omega_i = d_i \exp(\gamma_i x + \rho_i t)$. This method was successfully applied to the spatially homogeneous BE (Cornille and Gervois 1982, Cornille 1984), leading to the determination of a whole class of non-integrable equations sharing common properties (factorisation of the linear operator and bisoliton denominators without the soliton coupling term). Unfortunately the discrete Boltzmann models which are hyperbolic semi-linear equations, do not belong to that class. We must again investigate the class of possible bisolitons. The main difference between continuous and discrete BE is that for the second class the distributions themselves satisfy the linear conservation laws (we do not have to integrate over the velocity variable). Consequently these models are weakly non-linear. For instance for the 3-density Broadwell model (as well as for the 4-density one) the determination of the class of possible bisolitons is performed with the linear relations alone (see appendix 1). This possible class for 3 or 4 densities being the same as for 2, it follows that the non-integrable hyperbolic semi-linear equations define a particular class of non-linear equations with common properties.

In § 2 we study the solitons and the possible class of bisolitons. The solitons are self-similar solutions in the variable $\exp(\gamma x + \rho t)$ and represent planar shock waves. The full class of shock solutions is given and we recall that the particular Broadwell explicit soliton solution was an infinite-Mach-number shock wave. The bisolitons must be such that when one of the two soliton components is zero, then we recover the other soliton component $\omega_i = d_i \exp(\gamma_i x + \rho_i t)$. In appendix 1 trying denominators of the type $1+\Sigma\omega_i+\mu\omega_1\omega_2$, we find at the linear differential level of the 3- or 4-density models that only $\mu=1$ is not excluded. At this stage the bisoliton is only the sum of two solitons and the non-linear constraints will provide the coupling between them.

In § 3 we study the class of bisolitons with γ_i, ρ_i complex and conclude that positivity along the full x axis can be satisfied only with periodic solutions. The algebraic determination of these solutions is performed and sufficient positivity conditions obtained. They are planar waves propagating with the time but a strong absorption occurs. An analytic example is obtained for which the current J is asymptotically vanishing but the waves become non-propagating.

In § 4 we determine and discuss the positive solution along the full x axis (a larger class exists, positive along a semi-axis). They are written as a linear superposition of two solitons (or two planar shock waves) and a simple coupling condition between them is sufficient in order to satisfy the non-linear constraint of (1.1). Instead of a pure plane shock wave $x+ct$, $c=\rho/\gamma$ invariant by translation, they are a superposition of two plane shock waves $x+c_it$, $c=\rho_i/\gamma_i$ with a deformation of the shock profile when the time increases. At $t=0$ or at small t, they have the usual shock profiles, but when t is infinite they relax towards Maxwellian equilibrium states. These shocks are not permanent in time and vanish with large times.

In § 5 we introduce the mean free path ε into the collision term and look at the limits $\varepsilon\to 0$ (Caflish 1983). Here these limits are constant absolute Maxwellians. For the periodic solutions ($t\neq 0$) we find one limit, two for the planar shock waves and for the non-planar shock waves of § 4, we find three different limits which correspond to three different domains of the x, t plane. So for the present exact solutions these limits are non-uniform in x, t (except for the periodic solutions at $t\neq 0$). Further, there exist initial layers (at $t=0$ the solutions are ε dependent) and shock layers. There is no analytic ε expansion around $\varepsilon=0$, while a natural parameter for such an expansion seems to be $\exp(-1/\varepsilon)$.

2. 'Solitons' and possible 'bisolitons'

The solitons, solutions with one exponential variable, are easily deduced. They correspond to one-dimensional space in the variable $x+\text{constant}\times t$ and (1.1) becomes integrable. As a pedagogical example we quote the results in table 1 because the bisolitons will be studied in a similar way. Starting with the ansatz

$$V=v_0+v/\Delta \qquad W=w_0+w/\Delta \qquad Z=z_0+z/\Delta \tag{2.1}$$

and $\Delta=1+\omega$, $\omega=d\exp(\gamma x+\rho t)$; v, w, z being constants, we obtain the five relations of table 1(c). We define $y=v/w$ as a new parameter and the solitons (table 1(d)) depend on three parameters y, v_0, w_0 (the arbitrary constant d allowing a normalisation at $x=0$).

When $|x|\to\infty$, either $\Delta\to 1$ or $\Delta\to\infty$; in order to have $V>0$, we must satisfy both $v_0>0$ and $v_0+v>0$ (the same applies for W, Z). Consequently the mass $N=V+W+4Z$, when $|x|\to\infty$, has in general two different limits. These solutions represent planar shock waves without deformation of the profile when t is varying (look at a reference frame $x+t\rho/\gamma$). In table 1(e) we define the y intervals for which the positivity is satisfied and in table 1(f) some examples for which one of the asymptotic x limits corresponds to vanishing distributions (or an infinite shock).

Table 1. 'Solitons'.

(a) Ansatz
$V = v_0 + v/\Delta$, $W = w_0 + w/\Delta$, $Z = z_0 + z/\Delta$, $\Delta = 1 + d\exp(\gamma x + \rho t)$, $d > 0 \rightarrow \Delta \geq 1$; $v_0 \geq 0$ $w_0 \geq 0$ (not $v_0 = w_0 = 0$), $z_0 \geq 0$, v, w, z, ρ, γ real.

(b) Asymptotic positivity conditions
$\gamma x \rightarrow \infty$ or $\rho > 0$, $t \rightarrow \infty$, $\Delta \rightarrow \infty$, $(V\,W\,Z) \rightarrow (v_0\ w_0\ z_0)$
$\gamma x \rightarrow -\infty$ or $\rho < 0$, $t \rightarrow \infty$, $\Delta \rightarrow 1$, $(V\,W\,Z) \rightarrow (v_0 + v > 0\ \ w_0 + w > 0\ \ z_0 + z > 0)$
Asymp. pos. $\rightarrow$ positivity $\forall t \geq 0$ $\forall x \in R$ (due to $V\Delta = v_0\Delta + v \geq v_0 + v\ldots$).

(c) Relations
(1) $z_0 = \sqrt{v_0 w_0} > 0$, (2) $z(v + w) + vw = 0$, (3) $2\rho + v + w + z = 0$, (4) $\gamma(v + w) + \rho(v - w) = 0$, (5) $v_0 w + w_0 v + 2z(\rho - z_0) = 0$

(d) Algebraic solutions
def: $y = (v/w)$ $\tilde{y} = 1 + y + y^2 > 0$, $\bar{z} = z/w$ $\bar{\gamma} = \gamma/w$ $\bar{\rho} = \rho/w$ (2) $\bar{z} = -y/(1+y)$, (3) $2\bar{\rho} = -\tilde{y}/(1+y)$, (4) $2\bar{\gamma} = \tilde{y}(y-1)/(1+y)^2$, (5) $w = -(1+y)\,[(v_0 + w_0 y)(1+y) + 2z_0 y]/y\tilde{y}$. From v_0, w_0, y given $\rightarrow z_0$, w, v, z, γ, ρ.

(e) Physical solutions
$v_0 + v = -y v_0[1 + (w_0/v_0)^{1/2}(1+y)]^2/\tilde{y} > 0$ if $y < 0$, $w_0 + w = -w_0[y + (1+y)(v_0/w_0)^{1/2}]^2/y\tilde{y} > 0$ if $y < 0$, $z_0 + z = v_0[(1+y)(w_0/v_0)^{1/2} + 1]\{1 + y[(w_0/v_0)^{1/2} + 1]\}/\tilde{y} > 0$ either if $y < -1 - (v_0/w_0)^{1/2}$ or if $0 > y > -1/[1 + (w_0/v_0)^{1/2}]$. Ex: $w_0/v_0 = 1 \rightarrow y < -2$ or $-\frac{1}{2} < y < 0$, $w_0/v_0 = 4 \rightarrow y < -\frac{3}{2}$ or $-\frac{1}{3} < y < 0$, $v_0 = 0$ $y < -1$, $w_0 = 0$ $-1 < y < 0$.

(f) Simple examples
(1) $v_0 = z_0 = 0$, $w_0 > 0$, $y < -1$; $w = -w_0(1+y)^2/\tilde{y} < 0$, $v = wy > 0$, $z = w_0(1+y)y/\tilde{y} > 0$,
$2\rho = w_0(1+y) < 0$, $2\gamma = w_0(1-y) > 0$; $(VWZ) \xrightarrow[t \rightarrow \infty \text{ or } x \rightarrow -\infty]{} \left(v, \dfrac{-w_0 y}{\tilde{y}} > 0, z\right) \xrightarrow[x \rightarrow \infty]{} (0\ w_0\ 0)$.
(2) $w_0 = z_0 = 0$, $v_0 > 0$, $-1 < y < 0$; $w = -v_0(1+y)^2/y\tilde{y} > 0$, $v = wy < 0$, $z = v_0(1+y)/\tilde{y} > 0$,
$2\rho = v_0(1+y)/y < 0$, $2\gamma = v_0(1-y)/y < 0$; $(VWZ) \xrightarrow[t \rightarrow \infty \text{ or } x \rightarrow \infty]{} (-y v_0/\tilde{y} > 0, w, z) \xrightarrow[x \rightarrow -\infty]{} (v_0\ 0\ 0)$.

For instance we look at $w_0 = v_0 = 0$ (table 1(f), (2)) or an infinite-Mach shock at $x = -\infty$. Assuming further that at $+\infty$ the distributions V, W, Z are the same we find: $y = -\frac{1}{2}$, $\rho = -v_0/2$, $\gamma = -3v_0/2$. For the mass we find either v_0 at $-\infty$ or $4v_0$ at $+\infty$. This is the Broadwell shock solution.

To search for the possible bisolitons we introduce $\omega_i = d_i \exp(\gamma_i x + \rho_i t)$ and require $\rho_1\gamma_2 - \rho_2\gamma_1 \neq 0$ for a true two-dimensional solution. We prescribe that when $d_j = 0$, the bisoliton is reduced to the above soliton for d_i, $i = j$. The denominators must be of the type $\Delta = 1 + \Sigma\omega_i + \omega_1\omega_2 P(\omega_i, \omega_j)$, with P a polynomial. For simplicity we assume that P is a constant μ and substitute the ansatz (2.1) into (1.1) where now v, w, z are linear polynomials in ω_1, ω_2. In appendix 1 for the 3-density (1.1) model we require that such an ansatz satisfies the two linear conservation laws (1.2) and find that only $\mu = 1$ is not excluded or $\Delta = (1 + \omega_1)(1 + \omega_2)$. This result means (at the linear level of (1.1)) that the only possible bisolitons are a linear superposition of two solitons:

$$V = v_0 + \Sigma v_i/\Delta_i \qquad W = w_0 + \Sigma w_i/\Delta_i \qquad Z = z_0 + \Sigma z_i/\Delta_i \qquad \Delta_i = 1 + \omega_i \quad (2.2)$$

v_i, w_i, z_i being constants. The non-linear constraint of (1.1) will give the supplementary condition for the coupling of both solitons. We notice that the same result, $\mu = 1$, holds for the 4-density model (see appendix 1) with two conservation laws. For the 2-velocity model and the result $\mu = 1$ we must include a part of the non-linear constraint (Cornille 1987). In § 3 the soliton components are complex conjugate, and real in § 4. Consequently, in § 4 the solutions will represent generalisations of the planar shock waves while in § 3 they will have a different significance.

3. 'Bisolitons' with complex γ_i, ρ_i: periodic solutions

The bisolitons have denominators of the type $\Delta_i = 1 + d_i \exp(\gamma_i x + \rho_i t)$, $i = 1, 2$. We assume $\Delta_1 = \Delta_2^*$ and in a later stage $\mathrm{Re}\,\gamma_i = 0$ which will lead to periodic solutions.

We start with the ansatz

$$\begin{aligned} &W = w_0 + 2\,\mathrm{Re}\, w/\Delta \qquad && V = v_0 + 2\,\mathrm{Re}\, v/\Delta \qquad && Z = z_0 + 2\,\mathrm{Re}\, v/\Delta \\ &\Delta = 1 + d \exp(\gamma x + \rho t) && v_0, w_0, z_0 \text{ real} && v, w, z, \rho, \gamma \text{ complex} \end{aligned} \tag{3.1}$$

that we substitute into (1.1). Requiring that the coefficients of Δ^{-1}, Δ^{-2}, $|\Delta|^{-2}$ are zero, give six relations (see table 2(*b*)) in general complex, and ten real constraints among the thirteen real parameters v_0, w_0, z_0, v, w, z, ρ, γ. *A priori* the solutions depend on three arbitrary parameters. In addition $d = d_R + i d_I$ (which does not appear in table 2(*b*)) gives two other arbitrary parameters.

Table 2. 'Bisolitons' with complex γ_i, ρ_i.

(*a*) Ansatz
$V = v_0 + 2\,\mathrm{Re}(v/\Delta)$, $W = w_0 + 2\,\mathrm{Re}(w/\Delta)$, $Z_1 = z_0 + 2\,\mathrm{Re}(z/\Delta)$, $\Delta = 1 + d\exp(\gamma x + \rho t)$, $\gamma = \gamma_R + i\gamma_I$ same for ρ, v, w, z.

(*b*) Relations
(1) $z_0 = \pm\sqrt{v_0 w_0}$, (2) $|z|^2 = \mathrm{Re}(vw^*)$, (3) $(v+w)z + vw = 0$, (4) $2\rho + v + w + z = 0$, (5) $\gamma(w+v) + \rho(v-w) = 0$, (6) $v_0 w + w_0 v + 2z(\rho - z_0) = 0$.

(*c*) Algebraic solutions
def: $y\,\mathrm{e}^{\mathrm{i}\alpha} = v/w$, $\bar z = z/w$, $\bar\rho = \rho/w$, $\bar\gamma = \gamma/w$, $\bar z = \bar z_R + \mathrm{i}\bar z_I \ldots$, (2) $4\cos\alpha = -\lambda + \sqrt{\lambda^2 + 8}$, $\lambda = y + y^{-1}$, (3) $\bar z = -\cos\alpha(y + \mathrm{e}^{\mathrm{i}\alpha})$, (4) $2\bar\rho = \sin\alpha(\mathrm{e}^{\mathrm{i}\alpha} - y)\mathrm{i}$, (5) $\bar\gamma_R = \frac{1}{2}\sin^2\alpha\ (2\cos^2\alpha + \cos 2\alpha)$, $\bar\gamma_I = -\bar\gamma_R \bar z_R/\bar z_I$, (6) $v_0 + w_0 y\,\mathrm{e}^{\mathrm{i}\alpha} - 2z_0\bar z + 2w(A_0 + \mathrm{i}B_0) = 0$, $A_0 = \sin^2 2\alpha/4$, $B_0 = \sin 2\alpha(y^2 - \cos 2\alpha)/4$ (i) $\gamma_R \neq 0$; $w = w(y, v_0, w_0) \to z = \bar z w$, $\rho = \bar\rho w$, $\gamma = \bar\gamma w$; (ii) $\gamma_R = 0 \to z = z_R$ real, $w_I = w_R \bar\gamma_R/\bar\gamma_I$, (6′) $v_0 + w_0 y\,\mathrm{e}^{\mathrm{i}\alpha} - 2z_0\bar z + 2w_R(A_1 + \mathrm{i}B_1) = 0$, $2A_1\bar z_R = -y\cos\alpha\sin^2\alpha$, $B_1 = -A_1(1 + y(2\cos^2\alpha - \sin^2\alpha)/\cos\alpha)/y\sin\alpha$ (6″) $v_0 + w_0 A_2 = 2z_0 B_2$, $A_2 = \cos\alpha(1 + 2y\cos\alpha)/(\cos\alpha - y)$, $B_2 = (2\cos^2\alpha - 4y\sin^2\alpha\cos\alpha + y^2)/y(\cos\alpha - y)$ from (6″) $v_0/w_0 = (B_2 \pm \sqrt{B_2^2 - A_2})^2$, $\pm$ if $w_0 \gtrless 0$, $\to v_0(w_0, y)$; from (6′) $\to w_R$ and w_I, then $z = wz$, $\rho = w\bar\rho$, $\gamma = w\bar\gamma$.

(*d*) Asymptotic positivity
(i) $w_0 > 0$ and any y value: $\rho_R > 0$, $(V, W, Z)_{\mathrm{asymp}} = (v_0 > 0,\ w_0 > 0,\ z_0 > 0)$ ex: $y = 0.8$, $w_0 = 1$, $v_0 = 2.16$, $z_0 = 1.47$, $\rho = 3.7 - 0.89i$, $\gamma = -i2.6$, $v = -8.8 - i3.16$, $w = -6.15 + i4.9$, $z = 4.2$. (ii) $w_0 < 0$ and any y value: $\rho_R < 0$, $(V, W, Z)_{\mathrm{asymp}} = (v_0 + 2v_R > 0,\ w_0 + 2w_R > 0,\ z_0 + 2z_R > 0)$; $v_0 < 0$, $z_0 > 0$, $v_R > 0$, $w_R > 0$. ex: $y = 1.2$, $w_0 = -1$, $v_0 = -0.12$, $z_0 = 1.1$, $\rho = -0.41 - 0.008i$, $\gamma = 0.28i$, $v = 0.67 - i0.36$, $w = 0.61 + i0.52$.

3.1. *Determination of positive solutions (table 2)*

Notice that due to the relation (1) in table 2, w_0 and v_0 have the same sign. In order to build up the solutions we proceed in two successive steps: first we establish the algebraic solutions and second we take into account the positivity $V > 0$, $W > 0$, $Z > 0$ requirements for the physical solutions.

For the algebraic determination of the solutions (table 2(*c*)), it is convenient to introduce intermediate variables $y\,\mathrm{e}^{\mathrm{i}\alpha}$, $\bar z$, $\bar\rho$, $\bar\gamma$, ratios of v, z, ρ, γ by w and we choose y, w_0, v_0 as the arbitrary parameters. Then α, $\bar z$, $\bar\rho$, $\bar\gamma$ are y-dependent functions up to the relation (6) where two possibilities occur.

(i) We assume $\mathrm{Re}\,\gamma = \gamma_R \neq 0$ and the solutions are non-periodic. From any given v_0, w_0, y we find w and from the intermediate variables we reconstitute v, z, ρ, γ. Let us test the asymptotic positivity requirement. When $\mathrm{Re}\,\gamma x \to \pm\infty$, then either $(V, W, Z) \to (v_0, w_0, z_0)$ or $\to (v_0+2v_R, w_0+2w_R, z_0+2z_R)$ which must be positive quantities. From a numerical analysis we have not obtained positivity in both $\pm\infty$ sides. However, positive solutions exist (see an example in table 3(*b*)) on a semi-line (either $x>0$ or $x<0$). In the following we discard this case.

(ii) $\gamma_R = 0$ and we find that $z = z_R$ is real (see relations (5) and (ii) in table 2(*c*)). The solutions are periodic in x and depend *a priori* on two arbitrary parameters because we have introduced a new constraint. If in equation (6″) (table 2(*c*)) we take the square in both sides, then the equation becomes quadratic in v_0/w_0. This gives two determinations and we must check to what sign of $z_0 = \pm(v_0 w_0)^{1/2}$ they correspond. Coming back to (6′) and (6) we determine w and from the intermediate variables reconstruct all the quantities $z_0, w, v, z, \rho, \gamma$ as functions of y and w_0. Due to the quadratic non-linearity in (1) a first invariance property occurs. If w_0 is multiplied by a constant, then the same multiplying factor occurs for the other quantities so that without loss of generality we can restrict w_0 to the values 0, ± 1. Notice that from (2) only $\cos\alpha$ is determined and *a priori* the two possibilities $\pm\alpha$ must be distinguished. However there exists a second invariance property because the two opposite α values correspond to opposite values of the imaginary parts of the parameters w_I, v_I, ρ_I, γ_I.

Table 3. Analytical solution: $y = |v/w| = 1$.

(*a*) Periodic solutions

(*a*1) $v_0/w_0 = 1$, $w_0 > 0$

$$\begin{pmatrix} V \\ W \\ Z \end{pmatrix} = w_0 \begin{pmatrix} 1 - 2\,\mathrm{Re}\{[2+\sqrt{3} + \mathrm{i}(\sqrt{3}/2)^{1/2}(1+\sqrt{3})]/\Delta\} \\ 1 - 2\,\mathrm{Re}\{[2+\sqrt{3} - \mathrm{i}(\sqrt{3}/2)^{1/2}(1+\sqrt{3})]/\Delta\} \\ 1 + 2(1+\sqrt{3})\,\mathrm{Re}(1/\Delta) \end{pmatrix} \xrightarrow[t\to\infty]{} w_0 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\Delta = 1 + d \exp w_0\sqrt{3}\left(\frac{1+\sqrt{3}}{2}t - \mathrm{i}(\sqrt{3}/2)^{1/2}x\right) \xrightarrow[t\to\infty]{} \infty \qquad |d| > 12.3.$$

(*a*2) $v_0/w_0 = 1$, $w_0 < 0$

$$\begin{pmatrix} V \\ W \\ Z \end{pmatrix} = w_0 \begin{pmatrix} 1 - 2\,\mathrm{Re}\{[1/\sqrt{2} + \mathrm{i}(\sqrt{3}/2)^{1/2}((\sqrt{3}-1)/\sqrt{3})]/\Delta\} \\ 1 - 2\,\mathrm{Re}\{[1/\sqrt{2} - \mathrm{i}(\sqrt{3}/2)^{1/2}((\sqrt{3}-1)/\sqrt{3})]/\Delta\} \\ -1 + 2(1 - 1/\sqrt{3})\,\mathrm{Re}\,(1/\Delta) \end{pmatrix} \xrightarrow[t\to\infty]{} (-w_0)(2/\sqrt{3}-1) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\Delta = 1 + d \exp w_0\left[\frac{\sqrt{3}-1}{2}t + \mathrm{i}x\left(\frac{1-\sqrt{3}}{1+\sqrt{3}}\right)(\sqrt{3}/2)^{1/2}\right] \xrightarrow[t\to\infty]{} 1 \qquad |d| < 0.045.$$

(*b*) Positive solutions on a semi-line

$x \geqslant 0$, $\gamma_R > 0$, $\rho_R > 0$: $w_0 > 0$, $0 < y_0 = v_0/w_0 < 1$

$$\begin{pmatrix} V \\ W \\ Z \end{pmatrix} = w_0 \begin{pmatrix} y_0 - (2/\sqrt{3})\,\mathrm{Re}\{[1 + (1+\sqrt{3})(y_0+\sqrt{y_0}) + \mathrm{i}(\sqrt{3}/2)^{1/2}(1+\sqrt{3}+2\sqrt{y_0})]/\Delta\} \\ 1 - (2/\sqrt{3})\,\mathrm{Re}\{[(y_0 + (1+\sqrt{3})(1+\sqrt{y_0}) - \mathrm{i}(\sqrt{3}/2)^{1/2}(y_0(1+\sqrt{3})+2\sqrt{y_0})]/\Delta\} \\ \sqrt{y_0} + (2/\sqrt{3})\,\mathrm{Re}\{[2\sqrt{y_0} + (1+\sqrt{3})(1+y_0)/2 + \mathrm{i}(\sqrt{3}/2)^{1/2}(1-y_0)]/\Delta\} \end{pmatrix} \xrightarrow[t\to\infty]{} \begin{pmatrix} v_0 \\ w_0 \\ (v_0 w_0)^{1/2} \end{pmatrix}$$

$$\Delta = 1 + d\,\mathrm{e}^{\gamma x + \rho t} \quad \gamma/w_0 = \frac{(3-\sqrt{3})}{4}(1-y_0) - \frac{\mathrm{i}}{2}(\sqrt{3}/2)^{1/2}\left(1 + y_0 + \frac{4\sqrt{y_0}}{1+\sqrt{3}}\right)$$

$$\rho/w_0 = \sqrt{y_0} + (1+y_0)\left(\frac{1+\sqrt{3}}{4}\right) + \mathrm{i}(\sqrt{3}/2)^{1/2}(1-y_0)$$

Positivity $\forall t \geqslant 0$: Proposition 1 with $\alpha_3 = z \to |d| > 1 + 11/y_0$.

It follows that if we perform the same transformation for d_I, the imaginary part of d, then $\mathrm{Re}(v/\Delta)$ is unchanged (similarly for $\mathrm{Re}(w/\Delta)$, or z_R/Δ). For either $w_0=0$ or $v_0=0$ we have analytically proved from (6″)-(2) that no solution exists. Afterwards we consider y as the continuous parameter and $w_0=\pm 1$. In order to avoid $\Delta=0$, from $|1-|d|\exp\rho_R t|<|\Delta|$, we assume $|d|>1$ if $\rho_R>0$ ($|d|<1$ if $\rho_R<0$).

For the physical determination of the solutions we look first at the asymptotic positivity constraint (table 2(d)). If $\rho_R>0$ or $\rho_R<0$ we investigate $\lim_{t\to\infty}(V, W, Z)$ and find that either $|\Delta|\to\infty$ and we must have $v_0>0$, $w_0>0$ or $\Delta\to 1$ and necessarily $v_0+2v_R>0$, $w_0+2w_R>0$, $z_0+2z_R>0$. These two asymptotic states, either $v_0>0$, $w_0>0$, $z_0>0$ or $v_0+2v_R>0,\ldots$, are the corresponding Maxwellians of the discrete model where the velocities have fixed values. We obtain a first limitation on the class of possible solutions of table 2(c). Letting y be a continuous parameter, w_0 being positive or negative, we have numerically checked the signs of v_0, w_0, z_0, ρ_R, v_0+2v_R, w_0+2w_R, z_0+2z_R. The results are quoted in table 2(d). For $w_0>0$, $v_0>0$, and any value for y, we find that the acceptable physical solutions at $t=\infty$ have $\rho_R>0$, $z_0>0$ while for $w_0<0$, $v_0<0$, we find $z_0>0$, $\rho_R<0$ with the asymptotic constraint $v_0+2v_R>0,\ldots$, satisfied. In both cases, we notice that the acceptable solutions have $z_0=(v_0w_0)^{1/2}$ while the other determination $(-v_0w_0)^{1/2}$ is ruled out. Consequently for the quadratic equation deduced from (6″), only one of the two possible solutions is physically acceptable. As an illustration in table 1(d), we quote the numerical values for two examples $y=0.8$ and 1.2.

The last physical requirement is the positivity at $t=0$ and we introduce the two arbitrary integration constants $d=d_R+id_I$ which, until now, have not been discussed. Firstly we investigate the class $\rho_R>0$, $v_0>0$, $w_0>0$, $z_0>0$, discussing the positivity of the V density, the argument being the same for the other ones. Writing

$$\frac{V|\Delta|^2}{v_0}=1+2\frac{v_R}{v_0}+(|d|\,e^{\rho_R t})^2+2\,e^{\rho_R t}\,\mathrm{Re}\left[d\,e^{+i(\gamma_I x+\rho_I t)}\left(1+\frac{v^*}{v_0}\right)\right] \tag{3.2}$$

we notice that the RHS has a lower bound:

$$1-2\frac{|v_R|}{v_0}+(|d|\,e^{\rho_R t})^2-2|d|\,e^{\rho_R t}\left(1+\frac{|v|}{v_0}\right)$$

and deduce the following.

Proposition 1. For the class of solutions $w_0>0$, $v_0>0$, $z_0>0$, $\rho_R>0$, a sufficient condition in order that (V, W, Z) are positive densities is:

$$|d|>\sup(M_1, M_2, M_3) \qquad M_i=1+\frac{|\alpha_i|}{\alpha_{i0}}+\left[\left(1+\frac{|\alpha_i|}{\alpha_{i0}}\right)^2-1+2\frac{|\mathrm{Re}\,\alpha_i|}{\alpha_{i0}}\right]^{1/2}$$

$$\alpha_1=v \qquad \alpha_{10}=v_0$$

$$\alpha_2=w \qquad \alpha_{20}=w_0$$

$$\alpha_3=z_R \qquad \alpha_{30}=z_0.$$

Secondly we look at the second class of solutions $w_0<0$, $v_0<0$, $z_0>0$ and $\rho_R<0$. However we must treat V, W and Z differently. Starting with (3.2) written for either V, W or Z we deduce the two lower bounds:

(i)
$$V\frac{|\Delta|^2}{|v_0|}>-1+\frac{2v_R}{|v_0|}-(|d|\,e^{\rho_R t})^2-2\,e^{\rho_R t}|d|\left(1+\left|\frac{v}{v_0}\right|\right) \tag{3.3a}$$

and a similar one for $V \rightleftarrows W$,

(ii) $$Z\frac{|\Delta|^2}{z_0} > 1 + \frac{2z_R}{z_0} + (|d|\, e^{\rho_R t})^2 - 2\, e^{\rho_R t}|d|\left(1 + \frac{|z_R|}{z_0}\right). \tag{3.3b}$$

We recall that due to the asymptotic positivity ($t \to \infty$), the sum of the two first terms at the RHS of (3.3a, b) are positive. Then $|d|$ must be sufficiently small in order that the positivity of the RHS of (4a, b) be maintained.

Proposition 2. For the class of solutions $w_0 < 0$, $v_0 < 0$, $z_0 > 0$, $w_0 + 2w_R > 0$, $v_0 + 2v_R > 0$, $z_0 + 2z_R > 0$, $\rho_R < 0$, a sufficient condition in order that (V, W, Z) are positive densities is

$$|d| < \inf(N_1, N_2, P)$$

$$N_i = -\left(1 + \left|\frac{\beta_i}{\beta_{i0}}\right|\right) + \left[\left(1 + \left|\frac{\beta_i}{\beta_{i0}}\right|\right)^2 + \frac{2\,\mathrm{Re}\,\beta_i}{|\beta_{i0}|} - 1\right]^{1/2}$$

$$\beta_1 = v \qquad \beta_{10} = v_0 \qquad \beta_2 = w \qquad \beta_{20} = w_0$$

$$P = \left(1 + \frac{|z_R|}{z_0}\right) - \left[\left(1 + \frac{|z_R|}{z_0}\right)^2 - \left(1 + \frac{2z_R}{z_0}\right)\right]^{1/2}.$$

As an illustration, in figures 1(a, b), we plot the relaxation curves in the two cases $w_0 = 1$, $y = 0.8$, $d = 15(1+i)$ and $w_0 = -1$, $y = 1.2$, $d = 0.04(1+i)$, for which the numerical values of v, w, z, v_0, w_0, z_0, ρ, γ are given in table 2(d).

3.2. Analytical solution

There exists a simple case for which we can easily write down an analytical solution. If we start with $y = 1$, then $\cos\alpha = (-1 + \sqrt{3})/2$, in table 2($c$), $A_2 = -1$, $B_2 = 0$ leading to the simple solution $v_0 = w_0$. In table 3(a) we write down the solutions in both the $w_0 > 0$ and $w_0 < 0$ cases. The sufficient conditions on $|d|$, maintaining positivity for $t \geq 0$, have been calculated using the theoretical bounds of propositions 1 and 2. Constructing numerical solutions of table 3(a), we have verified that these constraints on $|d|$ are relevant. Let us notice that for this particular solution, $\rho_I = 0$ and the time dependence is real in Δ. The two possibilities $\pm\alpha$ correspond in table 3(a) to the changes $\pm(\frac{3}{2})^{1/4}$. If simultaneously we choose $d \to d^*$ then in final $v, w, \Delta \to v^*, w^*, \Delta^*$ and we have the same values for $\mathrm{Re}\, v/\Delta$, $\mathrm{Re}\, w/\Delta$.

3.3. Physical interpretation of the periodic solutions

We write down the total density $N = V + W + 4Z$ and current $J = V - W$ and look at their large time behaviour $N = N_{eq} + \delta N$, $J = J_{eq} + \delta J$. δN and δJ are small perturbations around the equilibrium states $N(t = \infty) = N_{eq}$, $J(t = \infty) = J_{eq}$. We notice that depending whether $\rho_R > 0$ or $\rho_R < 0$ we have $N_{eq} = v_0 + w_0 + 4z_0$, $J_{eq} = v_0 - w_0$ or $N_{eq} = v_0 + w_0 + 4z_0 + 2\,\mathrm{Re}(v + w + 4z)$, $J_{eq} = v_0 - w_0 + 2\,\mathrm{Re}(v - w)$ and find in both cases:

$$\begin{aligned} \delta_N &= 2A_N \exp(-|\rho_R| t) \cos(\gamma_I x + \rho_I t + \phi_N) \\ \delta_J &= 2A_J \exp(-|\rho_R| t) \cos(\gamma_I x + \rho_I t + \phi_J). \end{aligned} \tag{3.4}$$

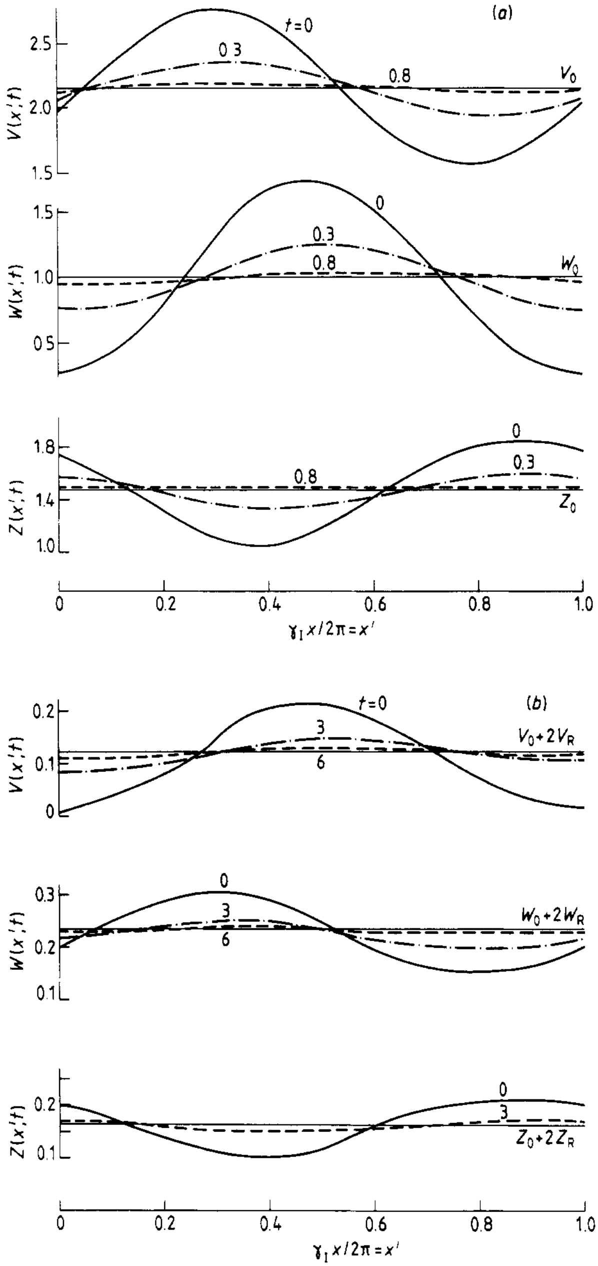

Figure 1. Plots of $V(x', t)$, $W(x', t)$, $Z(x', t)$ against $x' = \gamma_I x/2\pi$ for different t values, $x' \in [0, 1]$ corresponds to one period in the x variable. The numerical values of the parameters of the solutions are given in table 2(d): (i) and (ii).

(i) $(1-a)$: $w_0 = 1$, $y = |v/w| = 0.8$, $d = 15\,(1+i)$

(ii) $(1-b)$: $w_0 = -1$, $y = 1.2$, $d = 0.04(1+i)$.

A_N and A_J are positive constants and ϕ_N and ϕ_J are constant phase factors. Depending on whether $\rho_R > 0$ or $\rho_R < 0$, we have

$$\begin{aligned} &A_N \exp(i\phi_N) = (v+w+4z)/d \qquad A_J \exp(i\phi_J) = (v+w+4z)/d \qquad \rho_R > 0 \\ &A_N \exp(-i\phi_N) = -(v+w+4z)/d \qquad A_J \exp(-i\phi_J) = (w-v)d \qquad \rho_R < 0. \end{aligned} \tag{3.5}$$

Clearly δ_N and δ_J represent propagating ($\rho_I \neq 0$) and damped ($\rho_R \neq 0$) plane waves. Can they be compatible with damped sound waves? In that case we must have $J_{eq} = 0$ which means no transport of particle flux. Looking at the periodic solution we find that this is possible only if $v_0/w_0 = 1$ ($w_0 > 0$, $\rho_R > 0$ and $w_0 < 0$, $\rho_R < 0$). The corresponding periodic solution is the analytic one studied in § 3.2 (in table 3 we can verify in both cases that $J_{eq} = 0$). Unfortunately in that case $\rho_I = 0$ and the plane waves are non-propagating.

4. 'Bisolitons' with γ_i, ρ_i real: non-planar shock waves

We substitute the linear superposition of two (2.2) planar shock waves, $V = v_0 + \Sigma v_i/\Delta_i$, $W = \ldots$, $\Delta_i = 1 + \omega_i$, $\omega_i = d_i \exp(\gamma_i x + \rho_i t)$, into the system (1.1) and obtain (table 4(b)) for each soliton component the five soliton relations of table 1(c). A supplementary symmetric coupling soliton relation $2z_2z_1 = v_1w_2 + v_2w_1$ appears (from the vanishing of the $(\Delta_1\Delta_2)^{-1}$ coefficient) and represents the constraint, coming from the non-linear part, for the existence of a double plane shock wave.

Table 4. 'Bisolitons' with real γ_i, ρ_i.

(a) Ansatz
$V = v_0 + \Sigma v_i/\Delta_i$, $W = w_0 + \Sigma w_i/\Delta_i$, $Z = z_0 + \Sigma w_i/\Delta_i$, $\Delta = 1 + w_i$, $w_i = d_i \exp(\gamma_i x + \rho_i t)$, $d_i > 0$

(b) Relations
(1) $z_0^2 = v_0 w_0$, (2) $z_i(v_i + w_i) + v_i w_i = 0$, (3) $2\rho_i + v_i + w_i + z_i = 0$, (4) $\gamma_i(v_i + w_i) + \rho_i(v_i - w_i) = 0$, (5) $v_0 w_i + w_0 v_i + 2z_i(\rho_i - z_0) = 0$, (6) $2z_1z_2 - \Sigma v_i w_j = 0$.

(c) Algebraic solutions
def: $y_i = v_i/w_i$, $\tilde{y}_i = 1 + y_i + y_i^2$, $\bar{z}_i = z/w_i$, $\bar{\gamma}_i = \gamma/w_i$, $\bar{\rho}_i = \rho/w_i$, (2) $\bar{z}_i = -y_i/(1+y_i)$, (3) $2\bar{\rho}_i = -\tilde{y}_i/(1+y_i)$, (4) $2\bar{\gamma}_i = \tilde{y}_i(y_i - 1)/(1+y_i)^2$, (5) $w_i = -(1+y_i)[(v_0 + w_0 y_i)(1+y_i) + 2z_0 y_i]/y_i\tilde{y}_i$, (6) $y_j^2 + y_j(1+y_i^2)/(1+y_i) + y_i = 0$. From v_0, w_0, y_1 given $\rightarrow z_0$, y_2, $w_i \rightarrow v_i$, ρ_i, γ_i.

(d) Solution
$v_0 > 0$ $w_0 > 0$: $v_0 + v_i = -yv_0[1 + (w_0/v_0)^{1/2}(1+y_i)]^2/\tilde{y}_i > 0$ if $y_i < 0$, $w_0 + w_i = -w_0[y_i + (1+y_i)(w_0/v_0)^{1/2}]^2/y_i\tilde{y}_i > 0$ if $y_i < 0$, $z_0 + z_i = (v_0/\tilde{y}_i)[(1+y_i)(w_0/v_0)^{1/2} + 1] \times \{1 + y_i[(w_0/v_0)^{1/2} + 1]\} > 0$ either if $y_i < -[1 + (v_0/w_0)^{1/2}]$ or $0 > y_i > -1/[1 + (w_0/v_0)^{1/2}]$, $\gamma_i = w_0(1-y_i)\{y_i^2 + y_i[1 + (v_0/w_0)^{1/2}]^2 + v_0/w_0\}/2y_i(1+y_i)$, $\rho_i = \gamma_i(1+y_i)/(1-y_i)$.

4.1. *Algebraic determination and positivity of the solutions*

As in the soliton case, we define two new parameters $y_i = v_i/w_i$ and intermediate variables $\bar{z}_i$, $\bar{\rho}_i$, $\bar{\gamma}_i$ ratios of z_i, ρ_i, γ_i by w_i (table 4(c)). The original parameters v_i, w_i,

z_i, ρ_i, γ_i are deduced from the four y_i, v_0, w_0 ones. However y_1 and y_2 are linked by the above coupling relation (see figure 2(*a*)) leading to two determinations for y_2

$$y_2^{\pm}=[-(1+y_1^2)\pm\sqrt{D}]/2(1+y_1) \qquad D=(1-y_1)^4-12y_1^2 \tag{4.1}$$

and finally the bisoliton solutions depend on the three parameters v_0, w_0, y_1.

The general positivity discussion in terms of these three parameters is not simple. However, general considerations for the $|x|\to\infty$ limits are in order. We only have two possibilities.

(i) $\gamma_1\gamma_2<0$ and when $|x|\to\infty$, (V, W, Z) have the two limits $(v_0+v_i, w_0+w_i, z_0+z_i)$ $i=1,2$ which must be non-negative.

(ii) $\gamma_1\gamma_2>0$ and when $|x|\to\infty$, (V, W, Z) have the two limits (v_0, w_0, z_0) and $(v_0+\Sigma v_i, w_0+\Sigma w_i, z_0+\Sigma z_i)$ which must be non-negative.

For an analytical discussion, (i) is easier than (ii) because simple expressions exist for $v_0+v_i,\ldots$ (table 4(*d*)) but not for $v_0+\Sigma v_i,\ldots$. For the positivity it is unnecessary to discuss the different possible ρ_i signs. If the $|x|\to\infty$ positivity is satisfied, we can manage the d_i constants in ω_i such that the solutions remain positive for all x at $t=0$. Then the Broadwell system carries positivity along all $t>0$ values.

As an illustration we discuss the positivity for a simple (i) case $\gamma_1>0$, $\gamma_2<0$ with $v_0>0$, $w_0>0$, $z_0>0$. We want to obtain the conditions $v_0+v_i>0,\ldots$ and the corresponding domain into the y_1, v_0, w_0 space when $|x|\to\infty$. From the results $v_0+v_i>0$, $w_0+w_i>0$ we see $y_i<0$, $i=1,2$ and notice from figure 2(*a*) the restrictions on y_1 and y_2. We note that the signs of z_0+z_i and γ_i depend only on two parameters y_1 and v_0/w_0. Furthermore z_0+z_i and γ_1 have two and five sign changes respectively (table 4(*d*)). The ρ_i signs are obtained from $\rho_i=\gamma_i(1+y_i)/(1-y_i)$. In figure 2(*b*) we plot the v_0/w_0, y_1 domain such that $\gamma_1>0$, $\gamma_2<0$, $\rho_i>0$, $v_0+v_i>0,\ldots$ and for which the $|x|\to\infty$ positivity is satisfied. Noting that the coupling between y_1 and y_2 is symmetric,

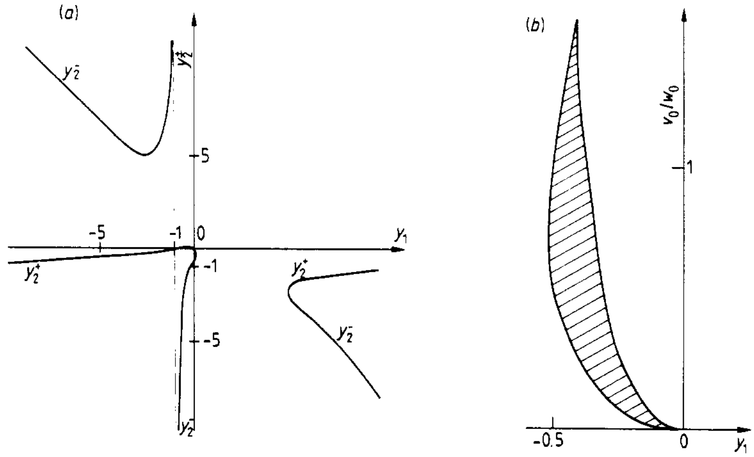

Figure 2. (*a*) Plot of the coupling relation y_2 against y_1 where

$$y_2^{\pm}=\frac{1}{2}\left(\frac{-(1+y_1^2)\pm\sqrt{D}}{1+y_1}\right)$$

$$D=(1-y_1)^4-12y_1^2.$$

(*b*) Plot of the $v_0/w_0, y_1$ domain for which $\gamma_1>0$, $\gamma_2<0$, $\rho_i>0$, $w_0+w_i>0$, $v_0+v_i>0$, $z_0+z_i>0$, $i=1,2$.

we find the solutions $\gamma_1<0$, $\gamma_2>0\ldots$ from the (figure 2(a)) domain with the change $y_1 \leftrightarrow y_2$. In order to obtain the positivity $\forall x$ at $t=0$ we rewrite Z

$$Z\Delta_1\Delta_2=(z_0+\Sigma z_i)+\sum_i (z_0+z_j)w_i+z_0 w_1 w_2 \qquad j\neq i \tag{4.2}$$

For the present case only the first term can be negative. On the RHS of (4.2) a trivial positive lower bound is obtained $\forall x>0$ or <0 using the fact that $\gamma_i\gamma_j<0$. For the $d_i>0$ constants a sufficient $Z>0$ condition is

$$d_i>-(z_0+\Sigma z_i)/(z_0+z_j) \qquad i\neq j. \tag{4.3}$$

For the positivity of V, W, we replace z_0, z_i by v_0, v_i, ... An example of such a solution is presented in figure 3 with $v_0/w_0=0.1$, $y_1=-0.2353$ (inside the figure 2(b) domain) and will be discussed below.

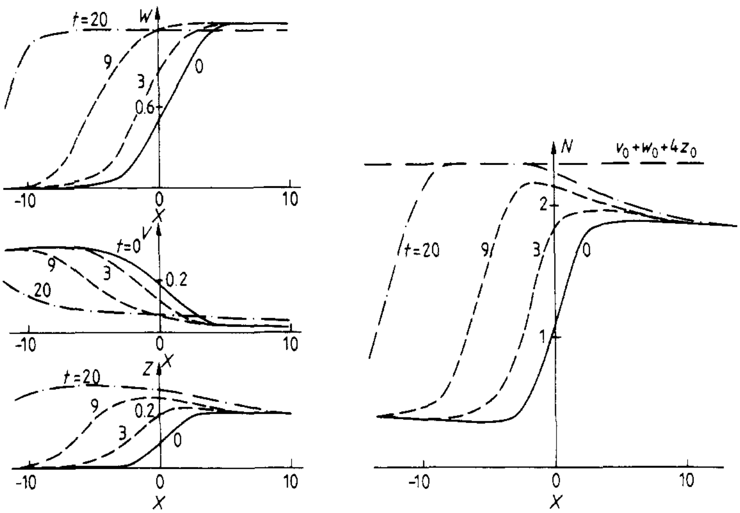

Figure 3. Non-planar shock waves V, W, Z, N against x: $v_0=1$, $w_0=0.1$, $y_1=-0.2303$, $\gamma_1=0.85$, $\gamma_2=-0.35$, $\rho_1=0.534$, $\rho_2=0.073$, $v_0+v_1=0.33$, $w_0+w_1=0.0009$, $z_0+z_1=0.017$, $v_0+v_2=0.035$, $w_0+w_2=1$, $z_0+z_2=0.19$.

As an illustration we discuss the positivity for a simple (ii) case: $w_0=0$, $v_0=0.1$, $z_0=\sqrt{0.1}$. Seeking $\gamma_1\gamma_2>0$ we find $y_1<-2.013$, and for these values $y_2=y_2^+<-0.375$, $\gamma_1>0$, $\gamma_2>0$, $\rho_1<0$, $\rho_2>0$. Further the positivity $|x|\to\infty$ is satisfied due to $v_0+\Sigma v_i>0$, $w_0+\ldots$, and when $t\to\infty$, the Maxwellians v_0+v_1, $w_0+\ldots$ are positive. Choosing an explicit example $y_1=-8.2$ we find: $v_1=6.37$, $w_1=-0.77$, $z_1=0.88$, $v_2=0.155$, $w_2=-0.195$, $z_2=-0.79$, $\gamma_1=4.14$, $\gamma_2=3.6$, $\rho_1=-3.2$, $\rho_2=0.404$ and with $d_1=d_2=1$, we have verified that the positivity for all x and $t\geqslant 0$ is satisfied. The physical interpretation is the same as in (i).

4.2. *Physical interpretation*

As in the soliton planar shock waves, for the bisoliton superposition of two such waves, the limits $x\to\pm\infty$ of V, W, Z are different. The mass $N=V+W+4Z$ has a jump between these two limits. At $t=0$ or t small, the profiles are very similar to the planar shock wave ones. However, when t increases and x remains fixed, a deformation of the profile occurs. The translation invariance $x+t\rho/\gamma$ disappears for a superposition $x_i+t\rho_i/\gamma_i$ of two planes (the velocities of the Broadwell model being ± 1, it follows that physically significant solutions must have $|\rho_i||\gamma_i|<1$). When the time is infinite and the space is finite then V, W, Z, N relax towards their respective constant Maxwellians. In figure 3 we present a numerical example of the deformations of the shock profiles and the relaxations towards equilibrium. For this example at small t and $x=-\infty$ we observe a strong shock with $w_0+w_1=0.0009$, $z_0+z_1=0.017$, $v_0+v_1=0.33$ and the Z, W density populations are negligible compared to the V one. In contrast, at $x=+\infty$, none of them is negligible. When t increases we observe for finite spatial values an extending plateau which becomes the equilibrium Maxwellian state when t is infinite.

4.3. *Multisolitons*

Can we have more than bisolitons? Starting (see table 4(b)) with $V=v_0+\Sigma v_i/\Delta_i$, $W=w_0+\ldots$, $i=1,\ldots,n$, $n\geq 2$, we find for each i the five soliton component relations plus the $n(n-1)/2$ coupling relations which, written with the $y_i=v_i/w_i$ parameters become: $y_i+y_j+y_i^2+y_j^2+y_iy_j(y_i+y_j)=0$. From the relations $\rho_i/\rho_j=(1+y_i)/(1-y_i)$ we see that the soliton components are different if the y_i are different. From the coupling relations and y_i real, this is possible only for $n=2$. As for the 2-velocity models we cannot have more than bisolitons.

5. Limits of the solutions when the mean free path goes to zero (Caflish 1983)

Let us define local Maxwellian (LM) solutions such that $Z_{\mathrm{LM}}^2-V_{\mathrm{LM}}W_{\mathrm{LM}}=0$, and the associated mass N_{LM} and current J_{LM}. From the identity $N_{\mathrm{LM}}+3(V_{\mathrm{LM}}+W_{\mathrm{LM}})=2(N_{\mathrm{LM}}^2+3J_{\mathrm{LM}}^2)^{1/2}$ we see that these LM satisfy the Euler equations: i.e. the mass conservation (1.2) and the conservation of momentum becoming: $J_t+\{N[2(1+3J^2/N^2)^{1/2}-1]/3\}_x=0$. Here the limits are constant absolute Maxwellians (AM) which satisfy trivially these equations. Note that for the periodic solutions we have one AM ($t\to\infty$), two AM for the planar shock wave ($x\to\mp\infty$) and three for the non-planar ones ($x\to\mp\infty$, $t\to\infty$).

We introduce the mean free path $\varepsilon>0$ into the collision term: $(Z^2-VW)/\varepsilon$ and remark that in (1.1) ε disappears with the change of variables: $t\to t/\varepsilon$, $x\to x/\varepsilon$. Let us call V_ε, W_ε, Z_ε, N_ε, J_ε the ε-dependent solutions. They are those determined in the previous sections with the change t/ε, x/ε. From this rescaling of t, x it follows that the positivity properties are the same. We discuss the limits of N_ε, J_ε when $\varepsilon>0\to 0$ and find that they are AM (we discuss only lim N_ε, since lim J_ε is then obvious).

(i) Periodic solutions. The change is: $\Delta_\varepsilon=1+d\exp(\rho_{\mathrm{R}}t/\varepsilon)\exp\mathrm{i}(\gamma_{\mathrm{I}}x+\rho_{\mathrm{I}}t)/\varepsilon$ and we find two possibilities for $t\neq 0$ (for $t=0$, the limit $\varepsilon\to 0$ does not exist). Either we have $\rho_{\mathrm{R}}>0$ and limit $1/\Delta_\varepsilon=0$, $N_{\mathrm{AM}}=v_0+w_0+4z_0$ or $\rho_{\mathrm{R}}<0$ and limit $1/\Delta_\varepsilon=1$, $N_{\mathrm{AM}}=Vv_0+w_0+z_0+2\,\mathrm{Re}(v+w+4z)$. For such solutions one limit exists: N_{AM}, J_{AM} which is an absolute Maxwellian.

(ii) Planar shock waves. The change is $\Delta_\varepsilon=1+d\exp(\gamma x+\rho t)/\varepsilon$ and we find two different limits for two different x, t subdomains of the half-plane x, t. Depending on whether $\gamma x+\rho t\gtreqless 0$ we find either $N_{\mathrm{AM}}=v_0+w_0+4z_0$ or $v_0+w_0+4z_0+v+w+4z$ which are AM.

(iii) Non-planar shock waves. The changes are $\Delta_{i\varepsilon} = 1 + d_i \exp(\gamma_i x + \rho_i t)/\varepsilon$. In the t, x half-plane the lines $\gamma_i x + \rho_i t = 0$ determine three different domains characterised by the signs of $\gamma_i x + \rho_i t$, $i = 1, 2$. Associated with these three domains exist three different limits which are AM.

As a first example we consider the class (i) of § 4.2 such that $\gamma_1 > 0$, $\rho_1 > 0$, $\gamma_2 < 0$, $\rho_1 > 0$. We define $\delta_i = \gamma_i x + \rho_i t$ and find

$$\delta_1 < 0,\ \delta_2 > 0 \qquad N_{\mathrm{AM}} = v_0 + w_0 + 4z_0 + v_1 + w_1 + 4z_1$$
$$\delta_1 > 0,\ \delta_2 > 0 \qquad N_{\mathrm{AM}} = v_0 + w_0 + 4z_0$$
$$\delta_1 > 0,\ \delta_2 < 0 \qquad N_{\mathrm{AM}} = v_0 + w_0 + 4z_0 + v_2 + w_2 + 4z_2.$$

For the second example (4.2, class (ii)) $\gamma_1 > 0$, $\rho_1 < 0$, $\gamma_2 > 0$, $\rho_2 > 0$ there also exist three different domains and limits.

In conclusion these limits $\varepsilon \to 0$ are non-uniform in x, t and ε appearing in terms like $\exp(\delta_i/\varepsilon)$, no analytic expansion exists around $\varepsilon = 0$ while a more natural parameter could be $\exp(-1/\varepsilon)$. In this paper we do not go further than these simple remarks.

6. Conclusion

For the one-spatial 2- and 3-density models, using the same algebraic method in both cases, we have obtained explicit two-dimensional solutions (beyond the known mathematical existence proofs) for a new class of non-integrable equations. The method works for the 4-density model (appendix) and presumably for all velocity models, however the positivity can be studied only case by case. It is remarkable that for spatial discrete mode (without boundary conditions, sources and sinks or external forces) positive exact two-dimensional solutions relaxing towards Maxwellians exist (in such a case no spatially exact solution is known for the continuous BE).

For the 2-velocity model, the most interesting exact solutions are the periodic damped sound waves while for the 3-density Broadwell model they are perhaps the two-dimensional non-planar shock waves which generalise the planar shock waves obtained by Broadwell more than twenty years ago.

However, in the future, the most interesting aspect of the discrete models may be the possibility of introducing more than one spatial dimension. If it is true that $1+1$ non-planar shock waves are more realistic than plane waves, I think that the important new step should be the determination of exact $(2+1)$-dimensional shock solutions.

After the completion of this work, I became aware of a recent determination by Golse (1985) of self-similar solutions (in the variable x/t) not studied here.

Acknowledgment

I thank Professor R Balian for his help in the physical interpretation of the solutions.

Appendix. Possible 'bisolitons' for the 3- and 4-density models

A1

Let us assume that three functions V, W, Z of the variables t, x, defined by

$$V = v_0 + v/\Delta \qquad W = w_0 + w/\Delta \qquad Z = z_0 + z/\Delta \qquad \Delta = 1 + \Sigma\omega_i + \mu\omega_1\omega_2$$

$$v = v_{00} + \Sigma\omega_i v_i \qquad w = w_{00} + \Sigma\omega_i w_i \qquad z = z_{00} + \Sigma\omega_i z_i \tag{A1}$$

$\omega_i = d_i \exp(\gamma_i x + \rho_i t)$

are (1+1)-dimensional solutions of two differential equations

$$(V-W)_t + (V+W)_x = 0 \qquad (V+\alpha Z)_t + (V+\beta Z)_x = 0 \qquad \alpha^2 \neq \beta^2 \tag{A2}$$

α, β being constants. We shall show that necessarily $\mu = 1$. We remark that in a two-dimensional space:

$$\rho_1\gamma_2 - \rho_2\gamma_1 \neq 0 \tag{A3}$$

and also that (A2) can be rewritten:

$$-v(\Delta_t + \Delta_x) + w(\Delta_t - \Delta_x) + \Delta(v_t + v_x - w_t + w_x) = 0 \tag{A4}$$

$$v(\Delta_t + \Delta_x) + z(\alpha\Delta_t + \beta\Delta_x) - \Delta(v_t + v_x + \alpha z_t + \beta z_x) = 0. \tag{A5}$$

A1.1

$\mu \neq 0, 1$: (A3) and (A4) are polynomials in ω_i and we require that the coefficients of ω_i, ω_i^2, $\omega_j\omega_i^2$, $\omega_1\omega_2$ are zero:

$$\begin{aligned} &v_i(\rho_j + \gamma_j) = w_i(\rho_j - \gamma_j) \qquad (\mu - 1)\Sigma(v_i(\rho_i + \gamma_i) + w_i(\gamma_i - \rho_i)) = 0 \\ &v_i(\rho_i - \gamma_i) = v_j(\rho_j - \gamma_j) \end{aligned} \tag{A4$'$}$$

$$\begin{aligned} &v_i(\rho_j + \gamma_j) + z_i(\alpha\rho_j + \beta\gamma_j) = 0 \qquad (\mu - 1)\Sigma(v_i(\rho_i + \gamma_i) + z_i(\beta\gamma_i + \alpha\rho_i)) = 0 \\ &v_i(\alpha\rho_i + \beta\gamma_i) = v_j(\alpha\rho_j + \beta\gamma_j). \end{aligned} \tag{A5$'$}$$

The last two relations (A4′) and (A5′) violate (A3).

A1.2

$\mu = 0$: Without loss of generality we can assume in (A1) $v_2 = w_2 = z_2 = 0$. Requiring in (A4) and (A5) that the coefficients of $\omega_1, \omega_2, \omega_1\omega_2, \omega_1^2$ are zero:

$$\begin{aligned} &w_{00}(\rho_2 - \gamma_2) = v_{00}(\rho_2 + \gamma_2) \qquad w_1(\rho_2 - \rho_1 + \gamma_1 - \gamma_2) = v_1(\rho_2 + \gamma_2 - \rho_1 - \gamma_1) \\ &v_1(\rho_2 - \gamma_2) = v_{00}(\rho_2 - \rho_1 + \gamma_1 - \gamma_2) \end{aligned} \tag{A4$''$}$$

$$\begin{aligned} &z_{00}(\alpha\rho_2 + \beta\gamma_2) + v_{00}(\rho_2 + \gamma_2) = 0 \\ &z_1(\alpha(\rho_2 - \rho_1) + \beta(\gamma_2 - \gamma_1)) + v_1(\rho_2 - \rho_1 + \gamma_2 - \gamma_1) = 0 \\ &v_1(\alpha\rho_2 + \beta\gamma_2) = v_{00}(\alpha(\rho_2 - \rho_1) + \beta(\gamma_2 - \gamma_1)). \end{aligned} \tag{A5$''$}$$

Still the two last relations (A4″) and (A5″) violate (A3). In conclusion only $\mu = 1$ is possible.

A2. Application to the 3-density Broadwell equation (1.1)

The two conservation laws (A2) are satisfied with $\alpha = 2, \beta = 0$.

A3. Application to the 4-density Broadwell models

There exist two such models. First the Broadwell model was studied by Tartar (1975) and Beale (1985)

$$U_{1t} + U_{1x} = U_{2t} - U_{2x} = -U_{3t} = -U_{4t} = bU_3U_4 - aU_1U_2. \tag{A6}$$

If we define $V = U_1$, $W = U_2$, $Z = U_3$, the conservation laws (A2) are satisfied with $\alpha = 1$, $\beta = 0$. Second the planar velocity model was studied by Gatignol (1975)

$$N_{0t} + N_{0x} = N_{3t} - N_{3x} = -2N_{1t} - N_{1x} = -2N_{2t} + N_{2x} = 2B(N_1 N_2 - N_0 N_3)/3. \qquad \text{(A7)}$$

If we define $V = N_0$, $W = N_3$, $Z = N_1$, the conservation laws (A2) are satisfied with $\alpha = 2$, $\beta = 1$.

In conclusion the only possible bisolitons for the one spatial 3- and 4-velocity Broadwell models are with $\mu = 1$, which means a superposition of two solitons.

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